# 1 Title

Log-shell and Log-volume

Kazumi Higashiyama

- Professor Mochizuki, [AbsTpIII], §3, §5
- Professor Hoshi, [Hsh], §1, §3

URL of this PDF: https://www.kurims.kyoto-u.ac.jp/[tilde]higashi/20210901-2.pdf

## 2 Abstract


Let $k$ be an MLF and $\bar{k}$ an algebraic closure of $k$. It is known that we can reconstruct a multiplicative group $\bar{k}^{\times}$ from an absolute Galois group $G_k$, despite we can not reconstruct the field structure of $\bar{k}$.

$$G_k \rightsquigarrow G_k \curvearrowright \bar{k}^{\times} : \text{a multiplicative group}, \qquad G_k \not\rightsquigarrow \bar{k} : \text{the field}$$

Therefore, we consider a new field $\bar{k}^{\sim}$ whose additive structure is derived from the multiplicative structure of $\bar{k}$, and we reconstruct the new field $\bar{k}^{\sim}$ from $\bar{k}$.

$$G_k \curvearrowright \bar{k} : \text{the field} \rightsquigarrow G_k \curvearrowright \bar{k}^{\sim} : \text{a field}$$

Then we define "log-volume" (notion essentially corresponding to Haar measure) and "log-shell" (the subgroup of $\bar{k}^{\sim}$ normalizing the log-volume). These are conventionally defined using an additive structure.

$$G_k \curvearrowright \bar{k} : \text{the field} \rightsquigarrow \text{a log-shell} \subseteq \bar{k}^{\sim}, \ \text{a log-volume}$$

Finally, we reconstruct an additive group $\bar{k}^{\sim}$, a log-shell, and a log-volume from $G_k$, despite we can not reconstruct the field structure of $\bar{k}^{\sim}$.

$$G_k \rightsquigarrow G_k \curvearrowright \bar{k}^{\sim} : \text{an additive group},$$

$$\rightsquigarrow \text{a log-shell} \subseteq \bar{k}^{\sim}, \ \text{a log-volume}$$

## 3 Notation

$k$: an MLF (finite extension of $\mathbb{Q}_{\exists p}$)

$\mathcal{O}_k$: the ring of integers of $k$

$m_k$: the maximal ideal of $\mathcal{O}_k$

In this talk, we use magenta objects and blue objects

Magenta: We start from $G_k$ (an abstract group)

Blue: We start from $G_k \curvearrowright \bar{k}$ (an abstract field)

## 4 Reconstruct Algorithms-1: Review

[AbsAnab], (1.2.1); [Hsh], (3.1)-(3.10)

START: $G_k$: an abstract group ($k$: MLF)

- $p$: the unique prime s.t. $\dim_{\mathbb{Q}_p}(G_k^{\mathrm{ab}} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p) \geq 2$
- $d \stackrel{\mathrm{def}}{=} \dim_{\mathbb{Q}_p}(G_k^{\mathrm{ab}} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_p) - 1$
- $p^f \stackrel{\mathrm{def}}{=} \sharp(G_k^{\mathrm{ab}})_{\mathrm{tor}}^{(p')} + 1$

Note that $d = [k : \mathbb{Q}_p]$, $f = [\mathcal{O}_k/m_k : \mathbb{F}_p]$

- $I_k \stackrel{\mathrm{def}}{=} \bigcap_{H \subseteq G_k} H$

where $H \subseteq G_k$: normal open s.t. $d(G_k)/f(G_k) = d(H)/f(H)$

Note that $I_k \subseteq G_k$: the inertia subgroup

- $P_k$: the unique pro-$p$ Sylow subgroup of $I_k$

Note that $P_k \subseteq I_k$: the wild inertia subgroup

## 5 Reconstruct Algorithms-2

- $\mathrm{Frob} \in G_k/I_k$: the unique element $G_k/I_k$ which acts on $I_k/P_k$ by $p^f$

Note that $\hat{\mathbb{Z}} \xrightarrow{\sim} G_k/I_k \colon 1 \mapsto \mathrm{Frob}$

- $k^\times \overset{\mathrm{def}}{=} G_k^{\mathrm{ab}} \times_{G_k/I_k} \mathrm{Frob}^{\mathbb{Z}} \subseteq G_k^{\mathrm{ab}}$
- $\mathcal{O}_k^\triangleright \overset{\mathrm{def}}{=} G_k^{\mathrm{ab}} \times_{G_k/I_k} \mathrm{Frob}^{\mathbb{N}} \subseteq k^\times$

Note that $\mathcal{O}_k^\triangleright = \mathcal{O}_k \setminus \{0\}$

- $\mathcal{O}_k^\times \overset{\mathrm{def}}{=} \mathrm{Im}(I_k \to G_k^{\mathrm{ab}}) \subseteq \mathcal{O}_k^\triangleright$
- $\bar{k}^\times \overset{\mathrm{def}}{=} \varinjlim_H k^\times(H)$
- $\mathcal{O}_{\bar{k}}^\triangleright \overset{\mathrm{def}}{=} \varinjlim_H \mathcal{O}^\triangleright(H)$
- $\mathcal{O}_{\bar{k}}^\times \overset{\mathrm{def}}{=} \varinjlim_H \mathcal{O}^\times(H)$

But we can not reconstruct the field structure of $k^\times, \bar{k}^\times$

<u>Proposition</u> ([AbsTpIII], (1.10))

If $k$: an MLF, $U/k$: a suitable curve, then

$$\pi_1(U) \rightsquigarrow \pi_1(U) \twoheadrightarrow G_k \rightsquigarrow k : \text{the field structure}$$

## 6 Motivation and Review of $p$-adic logarithm-1

For simplicity, now we consider $k = \mathbb{Q}_p$

- $\mu(\mathbb{Q}_p) \overset{\text{def}}{=} (\mathbb{Q}_p^\times)_{\text{tor}} \subseteq \mathbb{Q}_p$: the set of $(p-1)$-th roots of unity
- $\mathbb{Q}_p^\times = \{p^n u \mid n \in \mathbb{Z}, u \in \mathbb{Z}_p^\times\}$
- $\mathbb{Z}_p^\times = \mu(\mathbb{Q}_p) \times (1 + p\mathbb{Z}_p)$
- Then there exists a unique continuous group homomorphism

$$\log_{\mathbb{Q}_p} : \mathbb{Q}_p^\times \to \mathbb{Q}_p$$

s.t. $\log_{\mathbb{Q}_p}(p) = 0$, $\log_{\mathbb{Q}_p}(\mu(\mathbb{Q}_p)) = 0$,

$$\log_{\mathbb{Q}_p}(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots$$

for $1 + x \in 1 + p\mathbb{Z}_p$, and $\log_{\mathbb{Q}_p}$ induces

$$1 + p\mathbb{Z}_p \overset{\sim}{\to} p\mathbb{Z}_p \quad (\text{if } p \neq 2)$$

$$1 + p^2\mathbb{Z}_p \overset{\sim}{\to} p^2\mathbb{Z}_p$$

- In particular,

$$\frac{1}{2p} \log_{\mathbb{Q}_p}(\mathbb{Z}_p^\times) = \mathbb{Z}_p \quad (\text{if } p \neq 2)$$

$$\frac{1}{2p} \log_{\mathbb{Q}_p}(\mathbb{Z}_p^\times) \supseteq \mathbb{Z}_p \quad (\text{if } p = 2)$$

## 7 Motivation and Review of $p$-adic logarithm-2

We want to reconstruct a field $k$ or a ring $\mathcal{O}_k$ from $G_k$

So we consider the **log-shell**

$$\mathcal{I}_k \stackrel{\text{def}}{=} \frac{1}{2p} \log_k(\mathcal{O}_k^\times)$$

Note that $\mathcal{I}_k \supseteq \mathcal{O}_k$ (so $\mathcal{I}_k$ is a "container")

Thus,

$$G_k \overset{?}{\rightsquigarrow} k,\ \mathcal{O}_k,\ \mathcal{I}_k$$

$$G_k \curvearrowright \bar{k} \rightsquigarrow k,\ \mathcal{O}_k,\ \log_k\colon k^\times \to k \rightsquigarrow \mathcal{I}_k$$

So we consider next Lemma

## 8 Motivation and Review of $p$-adic logarithm-3

Key Lemma

We consider the injective limit of $p$-adic logarithm

$$\log_k \colon \mathcal{O}_k^\times \to k,$$

we obtain $p$-adic logarithm

$$\log_{\bar{k}} \colon \mathcal{O}_{\bar{k}}^\times \to \bar{k}$$

These induce the following group isomorphisms

$$k^\sim \stackrel{\text{def}}{=} (\mathcal{O}_k^\times)^{\text{pf}} \stackrel{\text{def}}{=} \varinjlim_{\mathcal{O}_k^\times \to \mathcal{O}_k^\times : N\text{-th power}} \mathcal{O}_k^\times \xrightarrow{\sim}_{\log_k} k$$

$$\bar{k}^\sim \stackrel{\text{def}}{=} (\mathcal{O}_{\bar{k}}^\times)^{\text{pf}} \stackrel{\text{def}}{=} \varinjlim_{\mathcal{O}_{\bar{k}}^\times \to \mathcal{O}_{\bar{k}}^\times : N\text{-th power}} \mathcal{O}_{\bar{k}}^\times \xrightarrow{\sim}_{\log_{\bar{k}}} \bar{k}$$

where

left side: multiplicative structure of $k$, $\bar{k}$

right side: additive structure of $k$, $\bar{k}$

Moreover, $\log_{\bar{k}}$ is compatible with the action $G_k \curvearrowright \bar{k}^\sim,\ \bar{k}$

## 9 Motivation and Review of $p$-adic logarithm-4

We consider $k^{\sim}$, $\bar{k}^{\sim}$ as fields

- additive structure of $k^{\sim}$, $\bar{k}^{\sim}$: induced by mult. structure of $k, \bar{k}$
- multiplicative structure of $k^{\sim}$, $\bar{k}^{\sim}$: inverse image $\log_{\bullet}^{-1}$ of mult. structure of $k, \bar{k}$

Thus, we obtain

$$G_k \rightsquigarrow \mathcal{O}_k^{\times} \rightsquigarrow k^{\sim} \text{ (additive group)}$$

$$G_k \curvearrowright \bar{k} \rightsquigarrow \log_{\bullet} \rightsquigarrow k^{\sim}, \ G_k \curvearrowright \bar{k}^{\sim} \text{(field)}$$

To reconstruct log-shell and log-volume, we use $k^{\sim}$ and we consider $\mathcal{O}_k^{\times} \to k^{\sim}$ instead of $\log_k \colon \mathcal{O}_k^{\times} \to k^{\sim} \to k$

## 10 Log-shell-1

START: $G_k \curvearrowright \bar{k}$: an abstract field

- $k \stackrel{\text{def}}{=} \bar{k}^{G_k}$: the fixed field
- $\mathbb{Z} \subseteq k$: the additive subgroup generated by $1 \in k$
- $\mathcal{O}_k \subseteq k, \mathcal{O}_{\bar{k}} \subseteq \bar{k}$: the integral closure of $\mathbb{Z}$
- $\mathcal{O}_k^\times \subseteq k, \mathcal{O}_{\bar{k}}^\times \subseteq \bar{k}$: the multiplicative groups

$$k^\sim \stackrel{\text{def}}{=} (\mathcal{O}_k^\times)^{\text{pf}} \stackrel{\text{def}}{=} \varinjlim_{\mathcal{O}_k^\times \to \mathcal{O}_k^\times : N\text{-th power}} \mathcal{O}_k^\times$$

$$\bar{k}^\sim \stackrel{\text{def}}{=} (\mathcal{O}_{\bar{k}}^\times)^{\text{pf}} \stackrel{\text{def}}{=} \varinjlim_{\mathcal{O}_{\bar{k}}^\times \to \mathcal{O}_{\bar{k}}^\times : N\text{-th power}} \mathcal{O}_{\bar{k}}^\times$$

## 11 Log-shell-2

Next, we consider $p$-adic logarithm (cf. p.6)

$$\log_k : \mathcal{O}_k^\times \to k$$

$$\log_{\bar{k}} : \mathcal{O}_{\bar{k}}^\times \to \bar{k}$$

$\log_\bullet$ induces

$$\mathcal{O}_k^\times \to k^\sim \to k$$

$$\mathcal{O}_{\bar{k}}^\times \to \bar{k}^\sim \to \bar{k}$$

## 12 Log-shell-3

By Key Lemma, $k^{\sim} \to k$, $\bar{k}^{\sim} \to \bar{k}$ are group isomorphisms

Since right sides are fields, $k^{\sim}, \bar{k}^{\sim}$ can be regarded as fields

- $k^{\sim}$ : a field
- $G_k \curvearrowright \bar{k}^{\sim}$ : a field
- $\mathcal{O}_{k^{\sim}}$: a ring of integers of $k^{\sim}$
- $\mathcal{I} \overset{\text{def}}{=} \frac{1}{2p}\mathrm{Im}(\mathcal{O}_k^{\times} \to k^{\sim}) \subseteq k^{\sim}$: an **log-shell**

<u>Lemma</u> ([Hsh], (1.2))

We consider $p$-adic logarithm

$$\mathcal{O}_k^{\times} \to k^{\sim} \to k$$

Then

$$\mathcal{I} \supseteq \mathcal{O}_{k^{\sim}} = (k^{\sim} \to k)^{-1}(\mathcal{O}_k)$$

that is, $\mathcal{I}$ is a "container"

## 13 Log-shell-4:blue version

START: $G_k \curvearrowright \bar{k}$: an abstract field ($k$: MLF)

$\rightsquigarrow$

- $k^{\sim}$: a field
- $G_k \curvearrowright \bar{k}^{\sim}$: a field
- $\log_k : \mathcal{O}_k^{\times} \to k^{\sim} \to k$: $p$-adic logarithm
- $\mathcal{O}_k \subseteq k$: a ring
- $\mathcal{O}_{k^{\sim}} = (k^{\sim} \to k)^{-1}(\mathcal{O}_k) \subseteq k^{\sim}$: a ring
- $\mathcal{I}_k \overset{\text{def}}{=} \frac{1}{2p} \log_k(\mathcal{O}_k^{\times}) \subseteq k$: a log-shell
- $\mathcal{I} \overset{\text{def}}{=} \frac{1}{2p} \mathrm{Im}(\mathcal{O}_k^{\times} \to k^{\sim}) \subseteq k^{\sim}$: a log-shell

## 14 Log-shell-5:magenta version

START: $G_k$: an abstract group ($k$: MLF)

$\rightsquigarrow$

◯ $\mathcal{O}_k^\times$: a multi. group

◯ $k^\sim$: an additive group

◯ $\bar{k}^\sim$: an additive group

× $\log_k : \mathcal{O}_k^\times \to k^\sim \to k$

◯ $\mathcal{O}_k^\times \to k^\sim$: instead of $p$-adic logarithm

× $\mathcal{O}_k \subseteq k$: a ring

× $\mathcal{O}_{k^\sim} \subseteq k^\sim$: a ring

× $\mathcal{I}_k \stackrel{\text{def}}{=} \frac{1}{2p} \log_k(\mathcal{O}_k^\times) \subseteq k$

◯ $\mathcal{I} \stackrel{\text{def}}{=} \frac{1}{2p} \text{Im}(\mathcal{O}_k^\times \to k^\sim) \subseteq k^\sim$: a log-shell

## 15 Log-volume-1

Theorem ([AbsTpIII], (5.7))

Write $\mathbb{M}(k) \stackrel{\text{def}}{=} \{\emptyset \neq U \subseteq k \mid U : \text{a compact open subset}\}$

Then there exists a unique map $\exists ! \mu_k \colon \mathbb{M}(k) \to \mathbb{R}_{>0}$ (Haar measure) which satisfies the following:

(1) **additivity**: $\forall A, B \in \mathbb{M}(k),\ A \cap B = \emptyset \implies \mu_k(A \cup B) = \mu_k(A) + \mu_k(B)$

(2) **⊞-translation invariance**: $\forall x \in k,\ \forall A \in \mathbb{M}(k),\ \mu_k(A + x) = \mu_k(A)$

(3) **normalization**: $\mu_k(\mathcal{O}_k) = 1$

We shall refer to $\mu_k(-)$ as the **volume** on $k$

and $\mu_k^{\log} \stackrel{\text{def}}{=} \log \circ \mu_k \colon \mathbb{M}(k) \to \mathbb{R}_{>0} \to \mathbb{R}$ as the **log-volume** on $k$,

where log dentes usual logarithm

## 16 Log-volume-2

START: $G_k \curvearrowright \bar{k}$

- $\mathbb{M}(k) \stackrel{\text{def}}{=} \{\emptyset \neq U \subseteq k \mid U : \text{a compact open subset}\}$
- $\mu_k : \mathbb{M}(k) \to \mathbb{R}_{>0}$: which satisfies (1), (2), (3)
- $\mu_k^{\log} : \mathbb{M}(k) \to \mathbb{R}$: the log-volume

Thus,

$$G_k \stackrel{?}{\rightsquigarrow} k,\ \mathcal{O}_k,\ \mu_k^{\log}$$

$$G_k \curvearrowright \bar{k} \rightsquigarrow k,\ \mathcal{O}_k \rightsquigarrow \mu_k^{\log}$$

So we consider $k^\sim$ and $\mathcal{I} \subseteq k^\sim$

## 17 Log-volume-3

<u>Theorem</u> ([Hsh], (3.12))

Write $\mathbb{M}(k^{\sim}) \overset{\text{def}}{=} \{\emptyset \neq U \subseteq k^{\sim} \mid U : \text{a compact open subset}\}$

Note that $\mathcal{I} \in \mathbb{M}(k^{\sim})$

Then there exists a unique Haar measure $\exists! \mu_{k^{\sim}} : \mathbb{M}(k^{\sim}) \to \mathbb{R}_{>0}$ which satisfies the following:

(1) **additivity**: $\forall A, B \in \mathbb{M}(k^{\sim}),\ A \cap B = \emptyset \implies \mu_{k^{\sim}}(A \cup B) = \mu_{k^{\sim}}(A) + \mu_{k^{\sim}}(B)$

(2) $\boxplus$**-translation invariance**: $\forall x \in k^{\sim},\ \forall A \in \mathbb{M}(k^{\sim}),\ \mu_{k^{\sim}}(A + x) = \mu_{k^{\sim}}(A)$

(3) **normalization**: $\mu_{k^{\sim}}(\mathcal{I}) = (p^*)^d / p^{f+m}$

where $p^m \overset{\text{def}}{=} \sharp \mu(k)^{(p)}$ and $\mu(k)^{(p)}$ denote the $p$-Sylow subgroup of $\mu(k)$

$$p^* \overset{\text{def}}{=} \begin{cases} p & (p : \text{odd}) \\ 4 & (p = 2) \end{cases}$$

Note that $\mu_{k^{\sim}}(\mathcal{O}_{k^{\sim}}) = \mu_k(\mathcal{O}_k) = 1$

## 18 Log-volume-4

START: $G_k$: an abstract group

We know $k^\times, p, d, f, k^\sim, \mathcal{I} \subseteq k^\sim$

- $p^* \stackrel{\text{def}}{=} p$ or $4$
- $\mu(k) \stackrel{\text{def}}{=} (k^\times)_{\text{tor}}$
- $\mu(k)^{(p)}$: the $p$-Sylow subgroup
- $p^m \stackrel{\text{def}}{=} \sharp\mu(k)^{(p)}$
- $\mathbb{M}(k^\sim) \stackrel{\text{def}}{=} \{\emptyset \neq U \subseteq k^\sim \mid U : \text{a compact open subset}\}$
- $\mu_{k^\sim} : \mathbb{M}(k^\sim) \to \mathbb{R}_{>0}$: which satisfies (1), (2), (3)
- $\mu_{k^\sim}^{\log} : \mathbb{M}(k^\sim) \to \mathbb{R}$: the log-volume

## 19 Summarize

We want to reconstruct

$$G_k \overset{?}{\rightsquigarrow} k,\ \mathcal{O}_k$$

Using the field structure, we can reconstruct

$$G_k \curvearrowright \bar{k} \rightsquigarrow k,\ \mathcal{O}_k \rightsquigarrow \mathcal{I}_k \subseteq k : \text{log-shell}, \quad \mu_k^{\log} : \text{log-volume}$$

By using the new field $k^{\sim}$, we can reconstruct

$$G_k \rightsquigarrow k^{\sim} : \text{additive structure} \rightsquigarrow \mathcal{I} \subseteq k^{\sim} : \text{log-shell}, \quad \mu_{k^{\sim}}^{\log} : \text{log-volume}$$

## References


[Hsh] Y. Hoshi, *Introduction to mono-anabelian geometry*, to appear in Proceedings of the conference “Fundamental Groups in Arithmetic Geometry”, Paris, France 2016.

[AbsAnab] S. Mochizuki, The Absolute Anabelian Geometry of Hyperbolic Curves, *Galois Theory and Modular Forms,* Kluwer Academic Publishers (2004), 77-122.

[AbsTpIII] S. Mochizuki, Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms, *J. Math. Sci. Univ. Tokyo.* **22** (2015), 939-1156.